Exercise 53

Find constants A and B such that the function $y = A\sin x + B\cos x$ satisfies the differential equation $y'' + y' - 2y = \sin x$.

Solution

Calculate the first derivative of y.

$$\begin{aligned} y' &= \frac{d}{dx}(A\sin x + B\cos x) \\ &= \frac{d}{dx}(A\sin x) + \frac{d}{dx}(B\cos x) \\ &= (A\cos x) + (-B\sin x) \\ &= A\cos x - B\sin x \end{aligned}$$

Calculate the second derivative of y.

$$\begin{aligned} y'' &= \frac{d}{dx}(A\cos x - B\sin x) \\ &= \frac{d}{dx}(A\cos x) - \frac{d}{dx}(B\sin x) \\ &= (-A\sin x) - (B\cos x) \\ &= -A\sin x - B\cos x \end{aligned}$$

Substitute these formulas into the differential equation.

$$y'' + y' - 2y = \sin x$$

$$(-A\sin x - B\cos x) + (A\cos x - B\sin x) - 2(A\sin x + B\cos x) = \sin x$$

Factor the sine and cosine terms.

$$(-A - B - 2A)\sin x + (-B + A - 2B)\cos x = \sin x$$

Match the coefficients of sine and cosine on both sides to obtain a system of equations for A and B.

$$-A - B - 2A = 1$$

$$-B + A - 2B = 0$$

Solve for A and B.

$$A = -\frac{3}{10} \qquad B = -\frac{1}{10}$$

Therefore,

$$y = -\frac{3}{10}\sin x - \frac{1}{10}\cos x.$$